

## Exact bound for histogram descriptors

In this supplementary material we derive an exact bound for the case of  $L_1$ -normalized histograms and  $\chi^2$  distance.

Given a window  $w$  we denote its  $L_1$ -normalized histogram by  $h^w = (\frac{h_1^w}{|w_1|}, \dots, \frac{h_{|Q|}^w}{|w_1|})$ , where  $Q$  is the set of bins considered and  $h_q^w$  represents the number of pixels that fall in bin  $q$  in window  $w$ . For two windows  $w_1$  and  $w_2$ , with corresponding  $L_1$ -normalized histograms  $h^{w_1}$  and  $h^{w_2}$ , their  $\chi^2$  distance is defined as:

$$\chi^2(h^{w_1}, h^{w_2}) = \sum_{q \in Q} \frac{\left(\frac{h_q^{w_1}}{|w_1|} - \frac{h_q^{w_2}}{|w_2|}\right)^2}{\frac{h_q^{w_1}}{|w_1|} + \frac{h_q^{w_2}}{|w_2|}} \quad (1)$$

For windows  $w_1$  and  $w_2$  with overlap  $o(w_1, w_2) = \frac{|w_1 \cap w_2|}{|w_1 \cup w_2|} = o$ , we want to derive the upper bound  $\mathcal{B}(w_1, w_2)$  such that  $\chi^2(h^{w_1}, h^{w_2}) \leq \mathcal{B}(w_1, w_2), \forall w_1, w_2$  with  $o(w_1, w_2) = o$ .

For simplicity of explanation, in the following we will use the term color histograms. However, our derivation holds for any histogram descriptors where the bins refer to any quantization of the feature space (e.g. bag-of-words with interest points as features).

**Theorem 1.** *Given two windows  $w_1$  and  $w_2$  with overlap  $o(w_1, w_2) = o$  we have:*

$$\chi^2(h^{w_1}, h^{w_2}) \leq \mathcal{B}(w_1, w_2) = \mathcal{B}(o) = 2 - 4 \cdot \frac{o}{o+1}, \forall w_1, w_2 \quad (2)$$

*Proof.* The  $\chi^2$  distance between two color histograms  $h^{w_1}$  and  $h^{w_2}$  is maximized when the two windows have no colors in common. In this case the  $\chi^2$  distance is 2. For overlapping windows  $w_1$  and  $w_2$ , the maximum  $\chi^2$  distance is reached for the case when the three regions  $w_1 \cap w_2$ ,  $w_1 \setminus w_2$  and  $w_2 \setminus w_1$  contain three disjoint sets of colors. Denoting with  $Q^w$  the set of colors that appear in window  $w$ , the upper bound  $\mathcal{B}(w_1, w_2)$  of the  $\chi^2$  distance is expressed as:

$$\mathcal{B}(w_1, w_2) = \sum_{q \in Q^{w_1 \setminus w_2}} \frac{\left(\frac{h_q^{w_1 \setminus w_2}}{|w_1|}\right)^2}{\frac{h_q^{w_1 \setminus w_2}}{|w_1|}} + \sum_{q \in Q^{w_2 \setminus w_1}} \frac{\left(\frac{h_q^{w_2 \setminus w_1}}{|w_2|}\right)^2}{\frac{h_q^{w_2 \setminus w_1}}{|w_2|}} + \sum_{q \in Q^{w_1 \cap w_2}} \frac{\left(\frac{h_q^{w_1 \cap w_2}}{|w_1|} - \frac{h_q^{w_1 \cap w_2}}{|w_2|}\right)^2}{\frac{h_q^{w_1 \cap w_2}}{|w_1|} + \frac{h_q^{w_1 \cap w_2}}{|w_2|}} \quad (3)$$

$$\mathcal{B}(w_1, w_2) = \sum_{q \in Q^{w_1 \setminus w_2}} \frac{h_q^{w_1 \setminus w_2}}{|w_1|} + \sum_{q \in Q^{w_2 \setminus w_1}} \frac{h_q^{w_2 \setminus w_1}}{|w_2|} + \sum_{q \in Q^{w_1 \cap w_2}} \frac{\left(\frac{h_q^{w_1 \cap w_2}}{|w_1|} - \frac{h_q^{w_1 \cap w_2}}{|w_2|}\right)^2}{\frac{h_q^{w_1 \cap w_2}}{|w_1|} + \frac{h_q^{w_1 \cap w_2}}{|w_2|}} \quad (4)$$

$$\mathcal{B}(w_1, w_2) = \frac{1}{|w_1|} \cdot \sum_{q \in Q^{w_1 \setminus w_2}} h_q^{w_1 \setminus w_2} + \frac{1}{|w_2|} \cdot \sum_{q \in Q^{w_2 \setminus w_1}} h_q^{w_2 \setminus w_1} + \left( \frac{\frac{1}{|w_1|} - \frac{1}{|w_2|}}{\frac{1}{|w_1|} + \frac{1}{|w_2|}} \right)^2 \cdot \sum_{q \in Q^{w_1 \cap w_2}} h_q^{w_1 \cap w_2} \quad (5)$$

$$\mathcal{B}(w_1, w_2) = \frac{|w_1 \setminus w_2|}{|w_1|} + \frac{|w_2 \setminus w_1|}{|w_2|} + |w_1 \cap w_2| \cdot \left( \frac{\frac{1}{|w_1|} - \frac{1}{|w_2|}}{\frac{1}{|w_1|} + \frac{1}{|w_2|}} \right)^2 \quad (6)$$

Given the overlap  $o = o(w_1, w_2)$  we distinguish two cases:

i)  $o = 0$ : This means that  $w_1 \cap w_2 = \emptyset$ , so  $w_1 \setminus w_2 = w_1$ ,  $w_2 \setminus w_1 = w_2$  which leads to

$$\mathcal{B}(w_1, w_2) = 1 + 1 + 0 = 2 \quad (7)$$

which in fact is the maximum  $\chi^2$  distance between two  $L_1$ -normalized histograms.

ii)  $o \neq 0$ : We make the following notations:

$$r_1 = \frac{|w_1 \cap w_2|}{|w_1|} \in (0, 1], r_2 = \frac{|w_1 \cap w_2|}{|w_2|} \in (0, 1], I = |w_1 \cap w_2| \quad (8)$$

Using these notations we express each of the terms from eq. (6) based on  $r_1, r_2, I$  as:

$$\frac{|w_1 \setminus w_2|}{|w_1|} = \frac{|w_1| - |w_1 \cap w_2|}{|w_1|} = 1 - r_1 \quad (9)$$

$$\frac{|w_2 \setminus w_1|}{|w_2|} = \frac{|w_2| - |w_2 \cap w_1|}{|w_2|} = 1 - r_2 \quad (10)$$

$$|w_1 \cap w_2| \cdot \frac{\left( \frac{1}{|w_1|} - \frac{1}{|w_2|} \right)^2}{\frac{1}{|w_1|} + \frac{1}{|w_2|}} = I \cdot \frac{\left( \frac{r_1}{I} - \frac{r_2}{I} \right)^2}{\frac{r_1}{I} + \frac{r_2}{I}} = \frac{(r_1 - r_2)^2}{r_1 + r_2} \quad (11)$$

Eq. (6) becomes

$$\mathcal{B}(w_1, w_2) = 1 - r_1 + 1 - r_2 + \frac{(r_1 - r_2)^2}{r_1 + r_2} \quad (12)$$

We can express the overlap  $o = o(w_1, w_2)$  using  $r_1, r_2, I$  as:

$$o = \frac{|w_1 \cap w_2|}{|w_1 \cup w_2|} = \frac{I}{\frac{I}{r_1} - I + \frac{I}{r_2} - I + I} = \frac{1}{\frac{1}{r_1} + \frac{1}{r_2} - 1} \quad (13)$$

so that we can write  $r_2$  as a function of  $r_1$  and  $o$ :

$$r_2 = \frac{o \cdot r_1}{o \cdot r_1 + r_1 - o} \quad (14)$$

and replace  $r_2$  from eq.(14) in eq.(12) in order we obtain:

$$\mathcal{B}(w_1, w_2) = 2 - r_1 - \frac{o \cdot r_1}{o \cdot r_1 + r_1 - o} + \frac{\left( r_1 - \frac{o \cdot r_1}{o \cdot r_1 + r_1 - o} \right)^2}{r_1 + \frac{o \cdot r_1}{o \cdot r_1 + r_1 - o}}$$

$$\begin{aligned}
\mathcal{B}(w_1, w_2) &= 2 - r_1 - \frac{o \cdot r_1}{o \cdot r_1 + r_1 - o} + \frac{(o \cdot r_1 + r_1 - 2 \cdot o)^2}{(o \cdot r_1 + r_1 - o) \cdot (o + 1)} \\
\mathcal{B}(w_1, w_2) &= 2 - r_1 + \frac{(o \cdot r_1 + r_1 - 2 \cdot o)^2 - o \cdot r_1 \cdot (o + 1)}{(o \cdot r_1 + r_1 - o) \cdot (o + 1)} \\
\mathcal{B}(w_1, w_2) &= 2 - r_1 + \frac{o^2 \cdot r_1^2 - 5 \cdot o^2 \cdot r_1 + 2 \cdot o \cdot r_1^2 + r_1^2 + 4 \cdot o^2 - 5 \cdot o \cdot r_1}{(o \cdot r_1 + r_1 - o) \cdot (o + 1)} \\
\mathcal{B}(w_1, w_2) &= 2 + \frac{4 \cdot o^2 - 4 \cdot o^2 \cdot r_1 - 4 \cdot o \cdot r_1}{(o \cdot r_1 + r_1 - o) \cdot (o + 1)} \\
\mathcal{B}(w_1, w_2) &= 2 - 4 \cdot o \cdot \frac{o \cdot r_1 - o + r_1}{(o \cdot r_1 + r_1 - o) \cdot (o + 1)} \\
\mathcal{B}(w_1, w_2) &= 2 - 4 \cdot \frac{o}{o + 1}
\end{aligned}$$

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